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Fedosov differentials and Catalan numbers*

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Abstract
The aim of the paper is to establish a non-recursive formula for the general solution of Fedosov's 'quadratic' fixed-point equation (Fedosov 1994 *J. Diff. Geom.* **40** 213–38). Fedosov's geometrical fixed-point equation for a differential is rewritten in a form similar to the functional equation for the generating function of Catalan numbers. This allows us to guess the solution. An adapted example for Kaehler manifolds of constant sectional curvature is considered in detail. Also for every connection on a manifold a familiar classical differential will be introduced.

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1. Introduction

1.1. Deformation quantization

Let (M, Π) be a Poisson manifold. The Poisson bracket $\{\cdot\,,\,\cdot\} : C^\infty(M)[[\lambda]] \times C^\infty(M)[[\lambda]] \to C^\infty(M)[[\lambda]]$, where λ is a formal parameter, is defined by

$$\{f, g\} := \Pi^{ik}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^k}.$$

The central definition in the theory of deformation quantization, established by Weyl and Moyal [14], Berezin [3, 4], and Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer [1, 2], is as follows.

A star product on (M, Π) is a binary operation $\star : C^\infty(M)[[\lambda]] \times C^\infty(M)[[\lambda]] \to C^\infty(M)[[\lambda]]$ with the following properties:

- $\star$ is a $\mathbb{C}[[\lambda]]$-bilinear associative product
- $f \star g = \sum_{l=0}^{\infty} \lambda^l C_l(f, g)$ with bi-differential operators C_l
- $1 \star f = f = f \star 1$
- $C_0(f, g) = fg$
- $C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}$

* Dedicated to the memory of Nikolai Neumaier.

A star product is continuous in the λ-adic topology.

Two star products on a Poisson manifold are called equivalent if there exists a formal series $E = id + \sum_{l=1}^{\infty} \lambda^l E_l$ of $\mathbb{C}[[\lambda]]$-linear differential operators $E_l : C^\infty(M)[[\lambda]] \to C^\infty(M)[[\lambda]]$ with $E(1) = 1$, $E_l(1) = 0$ for $l \geqslant 1$ and

$$f \star g = E^{-1}(Ef \star' Eg) \qquad \forall\ f, g \in C^\infty(M)[[\lambda]].$$

1.2. *Fedosov's quantization of symplectic manifolds*

Fedosov's geometric construction yields elegant, intrinsically defined formulae for all equivalence classes of star products on symplectic manifolds and is central for their classification.

According to Cattaneo, Felder and Tomassini the construction can also be adapted to globalize Kontsevitch's formality theorem [12] and construct a star product on any Poisson manifold [6].

The following sketch slightly differs from the one in [18] and Fedosov's notation and construction in [8].

1.2.1. The formal Weyl algebra The formal Weyl algebra $\mathcal{W} \otimes \Lambda^\bullet$ is central in Fedosov's quantization of a symplectic manifold (M, ω). It is defined by the complexified bundles

$$\mathcal{W} \otimes \Lambda^\bullet := \left(\prod_{s=0}^{\infty} \Gamma^\infty(\vee^s T^*M \otimes \Lambda^\bullet T^*M) \right)[[\lambda]].$$

The $\mathbb{C}[[\lambda]]$-linear degree maps $\deg_s, \deg_a, \deg_\lambda : \mathcal{W} \otimes \Lambda^\bullet \longrightarrow \mathcal{W} \otimes \Lambda^\bullet$ are defined in the usual way for a factorable tensor of the shape $S \otimes \alpha$ with $S \in \Gamma^\infty(\vee^s T^*M)$ and $\alpha \in \Gamma^\infty(\Lambda^k T^*M)$ by

$$\deg_s(S \otimes \alpha) := s(S \otimes \alpha)$$

$$\deg_a(S \otimes \alpha) := k(S \otimes \alpha)$$

$$\deg_\lambda(S \otimes \alpha) := \lambda \frac{\partial}{\partial \lambda}(S \otimes \alpha).$$

For $a \in \mathcal{W} \otimes \Lambda^k$ the notation $a = \sum_{l,s \in \mathbb{N}} \lambda^l a_l^s$ with $\deg_s a_l^s = s a_l^s$, $\deg_a a_l^s = k a_l^s$ and $\deg_\lambda a_l^s = 0$ is used. If $\deg_s a = 0$ the formal series notation $a = \sum_{l \in \mathbb{N}} \lambda^l a_l$ with $a_l \in \Gamma^\infty(\Lambda^k T^*M)$ and $\deg_\lambda a_l = 0$ is used.

The projection on the functions $C^\infty(M)[[\lambda]]$ is called σ.

For $a = f \otimes \alpha \in \mathcal{W} \otimes \Lambda^{k_1}$ and $b = g \otimes \beta \in \mathcal{W} \otimes \Lambda^{k_2}$ by

$$\mu(a \otimes b) := (f \vee g) \otimes (\alpha \wedge \beta),$$

an associative graded product μ is canonically defined and satisfies the property

$$\mu(a \otimes b) = (-1)^{k_1 k_2} \mu(b \otimes a),$$

called anti- or super commutativity. Also the notation $a \cdot b := \mu(a \otimes b)$ will be used.

A super derivation $\mathbf{D} : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet + d}$ of degree d is a linear map that satisfies the Leibniz rule

$$\mathbf{D}(a \cdot b) = (\mathbf{D}a) \cdot b + (-1)^{kd} a \cdot (\mathbf{D}b),$$

for $a \in \mathcal{W} \otimes \Lambda^k$. The vector space of super derivations together with the super commutator $[\ \cdot\ ,\ \cdot\ ]$, defined by

$$[\mathbf{D}_1, \mathbf{D}_2] := \mathbf{D}_1 \circ \mathbf{D}_2 - (-1)^{d_1 d_2} \mathbf{D}_2 \circ \mathbf{D}_1$$

for $\mathbf{D}_i : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet + d_i}$, is a super Lie algebra.

The degree maps $\deg_s, \deg_a, \deg_\lambda$ are super derivations with respect to μ.

Let $\delta : \mathcal{W} \otimes \Lambda^\bullet \longrightarrow \mathcal{W} \otimes \Lambda^{\bullet+1}$ and $\delta^* : \mathcal{W} \otimes \Lambda^\bullet \longrightarrow \mathcal{W} \otimes \Lambda^{\bullet-1}$ be defined in local coordinates by

$$\delta := (1 \otimes \mathrm{d}x^i) i_s(\partial_i) \quad \text{and} \quad \delta^* := (\mathrm{d}x^i \otimes 1) i_a(\partial_i),$$

where i_s and i_a are the symmetric and anti-symmetric contraction, respectively. The operators δ, δ^* are μ super derivations.

By

$$\delta^{-1} a := \begin{cases} \frac{1}{s+k}\delta^* a & \text{if} \quad \deg_s a = sa, \ \deg_a a = ka \ \text{ and } s+k \neq 0 \\ 0 & \text{if} \quad a \in C^\infty(M)[[\lambda]] \end{cases}$$

and linear continuation another operator $\delta^{-1} : \mathcal{W} \otimes \Lambda^\bullet \longrightarrow \mathcal{W} \otimes \Lambda^{\bullet-1}$ is defined.

Symmetry and grading arguments show that $\delta^2 = (\delta^*)^2 = (\delta^{-1})^2 = 0$ is valid. Moreover one has the following homotopy formula.

Lemma 1.2.1 (Poincaré lemma).

$$\delta\delta^{-1} + \delta^{-1}\delta + \sigma = id.$$

With the commuting symmetric insertion derivations the product μ can be deformed for every symplectic manifold in the Weyl–Moyal product, or in the case when (M, g) is a pseudo-Kaehler manifold, μ can be deformed in the Wick product as well [5, 16]. Let P be defined by

$$P := \frac{\mathrm{i}}{2}\Pi^{kl} i_s(\partial_k) \otimes i_s(\partial_l),$$

with the symplectic Poisson tensor Π in the Weyl case or

$$P := 2g^{k\bar{l}} i_s(\partial_{z^k}) \otimes i_s(\partial_{\bar{z}^l})$$

with the Kaehler metric g in the Wick case. The Weyl–Moyal product and the Wick product, respectively, are defined by

$$a \circ_\Pi b := \mu \circ \exp\big(\lambda P\big)(a \otimes b).$$

These products $\circ_\Pi$ are still graded, fiberwise and associative. Because of the shape of the Weyl–Moyal product the anti-symmetric degree-map $\deg_a$ and the so-called total degree map Deg defined by

$$\mathrm{Deg} := \deg_s + 2\deg_\lambda$$

are $\circ_\Pi$ derivations. The term of total degree k is $a^{(k)} = \sum_{s+2l=k} \lambda^l a_l^s$.

Let

$$\mathcal{W} \otimes \Lambda^\bullet = \mathcal{W}^{(0)} \otimes \Lambda^\bullet \supseteq \mathcal{W}^{(1)} \otimes \Lambda^\bullet \supseteq \mathcal{W}^{(2)} \otimes \Lambda^\bullet \supseteq \cdots \supseteq \{0\}$$

with $\bigcap_{d=0}^\infty \mathcal{W}^{(d)} \otimes \Lambda = \{0\}$, be the filtration corresponding to the total degree.

Since the symmetric insertion maps commute δ is a super derivation with respect to $\circ_\Pi$.

A bilinear super Lie bracket $[\,\cdot\,,\ \cdot\,]$ is defined by

$$[a, b] = a \circ_\Pi b - (-1)^{kl} b \circ_\Pi a,$$

for $a \in \mathcal{W} \otimes \Lambda^k$ and $b \in \mathcal{W} \otimes \Lambda^l$ and

$$[a, [b, c]] = [[a, b], c] + (-1)^{kl}[b, [a, c]]$$

is called the super Jakobi identity.

For every $a \in \mathcal{W} \otimes \Lambda^k$ the adjoint action of a in the Lie algebra, the so-called inner derivation $[a, \cdot\,]$ is defined by

$$[a, \cdot\,]b := [a, b].$$

Because $\circ_\Pi$ is associative and grading, the inner derivations satisfy

$$[a, \cdot\,](b \circ_\Pi c) = ([a, \cdot\,]b) \circ_\Pi c + (-1)^{kl} b \circ_\Pi ([a, \cdot\,]c)$$

and are $\circ_\Pi$ super derivations $\mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+k}$.

The so-called quasi inner derivations

$$\frac{\mathrm{i}}{\lambda}[a, \cdot\,]$$

are also well defined because of the super commutativity of μ. For example the super derivation δ can be written as a quasi inner derivation

$$-\delta = \frac{\mathrm{i}}{\lambda}[\delta^{-1}\omega, \cdot\,].$$

The equation $\left[\mathbf{D}, \frac{\mathrm{i}}{\lambda}[a, \cdot\,]\right] = \frac{\mathrm{i}}{\lambda}[\mathbf{D}a, \cdot\,]$ is valid if $\mathbf{D}$ is a $\circ_\Pi$ super derivation.

For a connection ∇ obviously ∇_{∂_i} is a μ derivation. The maps $\partial : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+1}$ and $\partial^* : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^\bullet$ defined by

$$\partial := (1 \otimes \mathrm{d}x^i)\nabla_{\partial_i} \quad \text{and} \quad \partial^* := (\mathrm{d}x^i \otimes 1)\nabla_{\partial_i}$$

are super derivations with respect to μ.

That the following equations are valid

$$[\delta, \partial^*] = \partial, \qquad [\delta^*, \partial] = \partial^*, \qquad [\delta, \partial] = 0, \qquad [\delta^*, \partial^*] = 0$$

can be shown with the tensor calculus.

A symplectic manifold (M, ω) always admits symplectic connections [13]. Because a symplectic connection is torsion-free and for $f \otimes \alpha \in \mathcal{W} \otimes \Lambda^\bullet$ the equation

$$\partial(f \otimes \alpha) = \nabla_{\partial_i} f \otimes \mathrm{d}x^i \wedge \alpha + f \otimes \mathrm{d}\alpha$$

is valid.

The Christoffel symbols of a symplectic connection satisfy

$$\frac{\partial \Pi^{kl}}{\partial x^i} + \Pi^{rl}\Gamma^k_{ir} + \Pi^{kr}\Gamma^l_{ir} = 0,$$

and from calculation it follows that ∇_{∂_i} is a $\circ_\Pi$ derivation. This fact yields that the map $\partial := (1 \otimes \mathrm{d}x^i)\nabla_{\partial_i}$, defined with a symplectic connection ∇, is a super derivation $\mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+1}$ with respect to $\circ_\Pi$.

1.2.2. Recursive flat differentials The first main step in Fedosov's construction is the existence of flat $\circ_\Pi$ super derivations $\mathcal{D} : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+1}$ of the shape $-\delta + \partial + \frac{\mathrm{i}}{\lambda}[\mathcal{R}, \cdot\,]$ with $\mathcal{R} \in \mathcal{W}^{(2)} \otimes \Lambda^1$. The condition that $(-\delta + \partial + \frac{\mathrm{i}}{\lambda}[\mathcal{R}, \cdot\,])^2$ vanishes is equivalent to the fact that $m := \partial + \frac{\mathrm{i}}{\lambda}[\mathcal{R}, \cdot\,]$ obeys the Maurer–Cartan equation

$$[-\delta, m] + \tfrac{1}{2}[m, m] = 0.$$

Because of

$$\partial^2 = \frac{\mathrm{i}}{\lambda}[R, \cdot\,]$$

where R is given by

$$R = \tfrac{1}{4}\omega_{kr} R^r_{lij}\, \mathrm{d}x^k \vee \mathrm{d}x^l \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j$$

in the Weyl case and

$$R = \frac{\mathrm{i}}{2} g_{k\bar{r}} R^{\bar{r}}_{\bar{l}i\bar{j}} \mathrm{d}z^k \vee \mathrm{d}\bar{z}^l \otimes \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j,$$

in the Wick case [5, 16], the curvature of the connection gets involved.

The equations

$$\delta R = 0 \quad \text{and} \quad \partial R = 0$$

are consequences of the Bianchi identities.

With the Deg-adic topology $\mathcal{W}^{(2)} \otimes \Lambda^1$ is a complete ultra-metric space.

Theorem 1.2.2. *For a formal series of closed two forms* $\Omega \in \lambda\Gamma^\infty(\Lambda^2 T^*M)[[\lambda]]$ *and every* $\mathcal{S} \in \mathcal{W}^{(3)} \otimes \Lambda^0$ *with* $\sigma(\mathcal{S}) = 0$ *there exists a unique element* $\mathcal{R}_{\Omega,\mathcal{S}} \in \mathcal{W}^{(2)} \otimes \Lambda^1$ *satisfying*

$$\delta\mathcal{R}_{\Omega,\mathcal{S}} = \Omega + R + \partial\mathcal{R}_{\Omega,\mathcal{S}} + \frac{\mathrm{i}}{\lambda}\mathcal{R}_{\Omega,\mathcal{S}} \circ_\Pi \mathcal{R}_{\Omega,\mathcal{S}} \quad \text{and} \quad \delta^{-1}\mathcal{R}_{\Omega,\mathcal{S}} = \mathcal{S}.$$

According to Banach's fixed-point theorem $\mathcal{R}_{\Omega,\mathcal{S}}$ *is the unique solution of the Fedosov fixed-point equation*

$$\mathcal{R}_{\Omega,\mathcal{S}} = \delta\mathcal{S} + \delta^{-1}\left(\Omega + R + \partial\mathcal{R}_{\Omega,\mathcal{S}} + \frac{\mathrm{i}}{\lambda}\mathcal{R}_{\Omega,\mathcal{S}} \circ_\Pi \mathcal{R}_{\Omega,\mathcal{S}}\right).$$

In this case the $\circ_\Pi$ *super derivation* $\mathcal{D}_{\Omega,\mathcal{S}} = -\delta + \partial + \frac{\mathrm{i}}{\lambda}[\mathcal{R}_{\Omega,\mathcal{S}}, \cdot\,]$ *is flat.*

Since Deg is a $\circ_\Pi$ derivation it is true that $\mathcal{R}_{\Omega,\mathcal{S}}$ satisfies the recursion

$$\mathcal{R}_{\Omega,\mathcal{S}}{}^{(2)} = \delta\mathcal{S}^{(3)}$$

and

$$\mathcal{R}^{(2+k)}_{\Omega,\mathcal{S}} = (\delta\mathcal{S} + \delta^{-1}(R + \Omega))^{(2+k)} + \delta^{-1}\left(\partial\mathcal{R}^{(1+k)}_{\Omega,\mathcal{S}} + \frac{\mathrm{i}}{\lambda}\sum_{l=1}^{k}\mathcal{R}^{(1+l)}_{\Omega,\mathcal{S}} \circ_\Pi \mathcal{R}^{(k+2-l)}_{\Omega,\mathcal{S}}\right)$$

for $k \geqslant 1$, but a non-recursive formula in this essential theorem is missing in [8].

Furthermore, an example of a non-zero curvature Fedosov construction is missing.

It is the intention of this work to take care and answer these two important issues in the following sections.

1.2.3. Fedosov star products Let

$$\mathcal{D}_{\Omega,\mathcal{S}} = -\delta + \partial + \frac{\mathrm{i}}{\lambda}[\mathcal{R}_{\Omega,\mathcal{S}}, \cdot\,]$$

be a flat $\circ_\Pi$ super derivation $\mathcal{D}_{\Omega,\mathcal{S}} : \mathcal{W}\otimes\Lambda^\bullet \to \mathcal{W}\otimes\Lambda^{\bullet+1}$ on the symplectic manifold (M, ω).

The map

$$\mathcal{D}^{-1}_{\Omega,\mathcal{S}} := -\frac{1}{1 - \left[\delta^{-1}, \partial + \frac{\mathrm{i}}{\lambda}[\mathcal{R}_{\Omega,\mathcal{S}}, \cdot\,]\right]}\delta^{-1}$$

is in the Deg-adic topology a well-defined endomorphism $\mathcal{D}^{-1}_{\Omega,\mathcal{S}} : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet-1}$ because $\left[\delta^{-1}, \partial + \frac{\mathrm{i}}{\lambda}[\mathcal{R}_{\Omega,\mathcal{S}}, \cdot\,]\right]$ is a linear contraction mapping.

Lemma 1.2.3 (Deformed Poincaré lemma).

$$\mathcal{D}^{-1}_{\Omega,\mathcal{S}}\mathcal{D}_{\Omega,\mathcal{S}} + \mathcal{D}_{\Omega,\mathcal{S}}\mathcal{D}^{-1}_{\Omega,\mathcal{S}} + \frac{1}{1 - \delta^{-1}\left(\partial + \frac{\mathrm{i}}{\lambda}[\mathcal{R}_{\Omega,\mathcal{S}}, \cdot\,]\right)}\sigma = id.$$

The deformed Poincaré lemma implies that the Fedosov–Taylor series

$$\tau_{\Omega,\mathcal{S}} : C^\infty(M)[[\lambda]] \to \mathcal{W} \otimes \Lambda^0 \cap ker\ \mathcal{D}_{\Omega,\mathcal{S}}$$

defined by

$$\tau_{\Omega,\mathcal{S}}(f) := \frac{1}{1-\delta^{-1}\big(\partial + \frac{\mathrm{i}}{\lambda}[\mathcal{R}_{\Omega,\mathcal{S}},\ \cdot\ ]\big)} f$$

is a $\mathbb{C}[[\lambda]]$-linear bijection with inverse σ.

Theorem 1.2.4. *On a symplectic manifold* (M, ω) *by*

$$f \star_{\nabla,\Omega,\mathcal{S}} g := \sigma(\tau_{\Omega,\mathcal{S}}(f) \circ_\Pi \tau_{\Omega,\mathcal{S}}(g))$$

star products are defined.

In the Weyl case $\star_{\nabla,\Omega,\mathcal{S}}$ satisfies

$$f \star_{\nabla,\Omega,\mathcal{S}} g = fg + \frac{\mathrm{i}\lambda}{2}\{f, g\} + O(\lambda^2),$$

and in the Wick case $\star_{\nabla,\Omega,\mathcal{S}}$ satisfies

$$f \star_{\nabla,\Omega,\mathcal{S}} g - g \star_{\nabla,\Omega,\mathcal{S}} f = \mathrm{i}\lambda\{f, g\} + O(\lambda^2).$$

Theorem 1.2.5 (Equivalence of Fedosov star products). *The two Fedosov star products* $\star_{\nabla,\Omega,\mathcal{S}}$ *and* $\star_{\nabla',\Omega',\mathcal{S}'}$ *on a symplectic manifold* (M, ω) *are equivalent if and only if*

$$[\Omega] = [\Omega'] \ \in \ \lambda H^2_{dR}(M, \mathbb{C})[[\lambda]].$$

Moreover Nest and Tsygan showed in [15] that every star product on a symplectic manifold is equivalent to a Fedosov star product, which substantiates the importance of the Fedosov construction in deformation quantization.

2. Non-recursive flat differentials

2.1. Non-recursive classical differentials

The projection on the classical part of a quantum Fedosov super derivation $\mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+1}$ is independent of $\Omega \in \lambda\Gamma^\infty(\Lambda^2 T^*M)[[\lambda]]$ and in the $\deg_s$-adic topology a well-defined flat μ super derivation [7].

On every manifold there also exists another natural μ super differential $\mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+1}$ of familiar shape in $-1 \leqslant \deg_s \leqslant 0$.

Theorem 2.1.1. *In the* $\deg_s$*-adic topology the map* $\mathcal{D} : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+1}$ *defined by*

$$\mathcal{D} := -\exp(\partial^*)\delta \exp(-\partial^*)$$

is a well-defined flat μ *super derivation.*

Proof. Sorting of $\mathcal{D}$ in the symmetric degree shows $\mathcal{D} = \sum_{s=-1}^{\infty} \mathcal{D}^s$, where the super derivations $\mathcal{D}^s = -\frac{1}{(s+1)!}\sum_{h=0}^{s+1}(-1)^h\binom{s+1}{h}(\partial^*)^{s+1-h}\delta(\partial^*)^h : \mathcal{W} \otimes \Lambda^\bullet \to \mathcal{W} \otimes \Lambda^{\bullet+1}$ satisfy the recursion $\mathcal{D}^{-1} = -\delta$ and $\mathcal{D}^s = \frac{1}{s+1}[\partial^*, \mathcal{D}^{s-1}] \quad \forall s \geqslant 0$.

That $\mathcal{D}$ squares to zero is easy to see because of $\delta^2 = 0$ and the exponential addition theorem.

Calculation shows

$$\mathcal{D} = -(1 \otimes \mathrm{d}x^i)i_s(\partial_i) + (1 \otimes \mathrm{d}x^i)\nabla_{\partial_i} - \tfrac{1}{2}(\mathrm{d}x^j \otimes \mathrm{d}x^i)(R(\partial_i, \partial_j) - T^m_{ij}\nabla_{\partial_m}) + \{\deg_s \geqslant 2\}.$$

□

2.2. Non-recursive Fedosov differentials

Applying the geometrical series $(1-\delta^{-1}\partial)^{-1}$ on the Fedosov fixed-point equation

$$\mathcal{R}_{\Omega,\mathcal{S}} = \delta\mathcal{S} + \delta^{-1}(\Omega + R) + \delta^{-1}\partial\mathcal{R}_{\Omega,\mathcal{S}} + \frac{\mathrm{i}}{2\lambda}\delta^{-1}[\mathcal{R}_{\Omega,\mathcal{S}}, \mathcal{R}_{\Omega,\mathcal{S}}]$$

shows a new 'quadratic' fixed-point equation

$$\mathcal{R}_{\Omega,\mathcal{S}} = \mathbf{R}_{\Omega,\mathcal{S}} + \mathcal{R}_{\Omega,\mathcal{S}}\Diamond\mathcal{R}_{\Omega,\mathcal{S}}$$

where

$$\mathbf{R}_{\Omega,\mathcal{S}} := \frac{1}{1-\delta^{-1}\partial}(\delta\mathcal{S} + \delta^{-1}(\Omega + R))$$

is the inhomogeneity and the non-fiberwise binary operation $\Diamond : (\mathcal{W}\otimes\Lambda^k)\times(\mathcal{W}\otimes\Lambda^l) \to \mathcal{W}\otimes\Lambda^{k+l-1}$ defined by

$$\Diamond := \frac{\mathrm{i}}{2\lambda}\frac{1}{1-\delta^{-1}\partial}\delta^{-1}[\,\cdot\,,\,\cdot\,]$$

replaces the fiberwise binary operation $\frac{\mathrm{i}}{2\lambda}\delta^{-1}[\,\cdot\,,\,\cdot\,]$.

The two fixed-point equations are equivalent, in the sense that they define the same unique fixed point but the linear term $\delta^{-1}\partial$ in the contraction mapping, defining the unique fixed point, is canceled.

The bilinear binary operation $\Diamond : (\mathcal{W}\otimes\Lambda^1)\times(\mathcal{W}\otimes\Lambda^1) \to \mathcal{W}\otimes\Lambda^1$ is commutative but non-associative. The next well-known lemma explains why the new fixed-point equation for $\mathcal{R}_{\Omega,\mathcal{S}}$ should be called the standard 'quadratic' fixed-point equation.

Lemma 2.2.1 (Parentheses and Catalan numbers). *There exist*

$$C_n := \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1-\sqrt{1-4z}}{2}\right)(0)$$

different ways in which a sequence of n factors a_i and $n-1$ binary operations $\Diamond_i$ can be well parenthesized.

The reason for this lemma is that the generating function of the Catalan numbers $C(z) := \sum_{n=0}^{\infty} C_n z^n$ satisfies $C(z) = z + C^2(z)$ because of the Cauchy product and the recursion $C_0 = 0$, $C_1 = 1$ and $C_n = \sum_{l=1}^{n-1} C_l C_{n-l}\ \forall\, n \geqslant 2$. The definition as a formal series and $C_0 = 0$ result in $C(z) = (1-\sqrt{1-4z})/2$.

The equation $C_n = \frac{1}{n}\binom{2n-2}{n-1}\ \forall\, n \in \mathbb{N}^+$ is well known.

The quadratic equation for the generating function of Catalan numbers is central to the solution of Fedosov's fixed-point equation with a similar structure.

To distinguish all different parentheses numbering them by $(a_1\Diamond_1\cdots\Diamond_{n-1}a_n)_p$ or just $(a)_p^{\Diamond n}$ if $a_i = a$ and $\Diamond_i = \Diamond$ with $1 \leqslant p \leqslant C_n$ makes sense.

Theorem 2.2.2. *The Fedosov derivation $\mathcal{D}_\Omega$ is*

$$\mathcal{D}_\Omega = -\delta + \partial + \frac{\mathrm{i}}{\lambda}\left[\sum_{n=1}^{\infty}\sum_{p=1}^{C_n}\left(\frac{1}{1-\delta^{-1}\partial}\delta^{-1}(\Omega + R)\right)_p^{(\frac{\mathrm{i}}{2\lambda}\frac{1}{1-\delta^{-1}\partial}\delta^{-1}[\,\cdot\,,\,\cdot\,])n}\ ,\ \cdot\ \right]$$

Proof. The unique fixed point $\mathcal{R}_{\Omega,\mathcal{S}}$ of the standard 'quadratic' contraction mapping $Q : \mathcal{W}^{(2)}\otimes\Lambda^1 \to \mathcal{W}^{(2)}\otimes\Lambda^1$ defined by

$$Q(a) := \mathbf{R}_{\Omega,\mathcal{S}} + (a)_1^{\Diamond 2}$$

is required.

By degree counting

$$\lozenge : (\mathcal{W}^{(2+n_l)} \otimes \Lambda^1) \times (\mathcal{W}^{(2+n_r)} \otimes \Lambda^1) \to \mathcal{W}^{(3+n_l+n_r)} \otimes \Lambda^1,$$

and

$$(\mathbf{R}_{\Omega,\mathcal{S}})_p^{\lozenge(1+n)} \in \mathcal{W}^{(2+n)} \otimes \Lambda^1$$

is obvious.

Induction over N shows that the Banach fixed-point iteration is

$$Q^{N+1}(0) = \mathbf{R}_{\Omega,\mathcal{S}} + \underbrace{(\mathbf{R}_{\Omega,\mathcal{S}} + (\mathbf{R}_{\Omega,\mathcal{S}} + (\ldots (\mathbf{R}_{\Omega,\mathcal{S}} + (}_{N} \mathbf{R}_{\Omega,\mathcal{S}})_1^{\lozenge 2})_1^{\lozenge 2} \ldots)_1^{\lozenge 2})_1^{\lozenge 2})_1^{\lozenge 2}.$$

In this non-associative setting the right suggestion is to take all different parentheses.

Because of the Catalan recursion it is in some sense a universal formula that by $\mathcal{R} := \sum_{n=1}^{\infty} \sum_{p=1}^{C_n} (\mathbf{R})_p^{\lozenge n}$, if it is convergent, a fixed point $\mathcal{R}$ of a standard 'quadratic' fixed-point equation of the form $\mathcal{R} = \mathbf{R} + (\mathcal{R})_1^{\lozenge 2}$ is defined. The following calculation shows that the unique fixed point $\mathcal{R}_{\Omega,\mathcal{S}} \in \mathcal{W}^{(2)} \otimes \Lambda^1$ of the Fedosov fixed-point equation is

$$\mathcal{R}_{\Omega,\mathcal{S}} := \sum_{n=1}^{\infty} \sum_{p=1}^{C_n} \left(\frac{1}{1-\delta^{-1}\partial} (\delta\mathcal{S} + \delta^{-1}(\Omega + R)) \right)_p^{(\frac{\mathrm{i}}{2\lambda} \frac{1}{1-\delta^{-1}\partial} \delta^{-1}[\,\cdot\,,\,\cdot\,])n}.$$

First of all $\lim_{N\to\infty} \sum_{n=1}^{N} \sum_{p=1}^{C_n} (\mathbf{R}_{\Omega,\mathcal{S}})_p^{\lozenge n} \in \mathcal{W}^{(2)} \otimes \Lambda^1$ is clearly convergent and well defined in the Deg-adic topology:

$$Q(\mathcal{R}_{\Omega,\mathcal{S}}) = Q\left(\lim_{N\to\infty} \sum_{n=1}^{N} \sum_{p=1}^{C_n} (\mathbf{R}_{\Omega,\mathcal{S}})_p^{\lozenge n} \right) = \lim_{N\to\infty} Q\left(\sum_{n=1}^{N} \sum_{p=1}^{C_n} (\mathbf{R}_{\Omega,\mathcal{S}})_p^{\lozenge n} \right)$$

$$= (\mathbf{R}_{\Omega,\mathcal{S}})_1^{\lozenge 1} + \lim_{N\to\infty} \left\{ \sum_{\substack{n_{l,r}\in\mathbb{N}^+ \\ n_l+n_r\leqslant N+1 \\ 1\leqslant p_{l,r}\leqslant C_{n_{l,r}}}} (\mathbf{R}_{\Omega,\mathcal{S}})_{p_l}^{\lozenge n_l} \lozenge (\mathbf{R}_{\Omega,\mathcal{S}})_{p_r}^{\lozenge n_r} + \sum_{\substack{1\leqslant n_{l,r}\leqslant N \\ n_l+n_r\geqslant N+2 \\ 1\leqslant p_{l,r}\leqslant C_{n_{l,r}}}} (\mathbf{R}_{\Omega,\mathcal{S}})_{p_l}^{\lozenge n_l} \lozenge (\mathbf{R}_{\Omega,\mathcal{S}})_{p_r}^{\lozenge n_r} \right\}.$$

Because of

$$(\mathbf{R}_{\Omega,\mathcal{S}})_p^{\lozenge(2+N)} \in \mathcal{W}^{(3+N)} \otimes \Lambda^1$$

the last term converges to zero in the Deg-adic topology and the calculation results in

$$Q(\mathcal{R}_{\Omega,\mathcal{S}}) = (\mathbf{R}_{\Omega,\mathcal{S}})_1^{\lozenge 1} + \sum_{\substack{n_{l,r}\in\mathbb{N}^+,\, n\geqslant 2 \\ n_l+n_r=n \\ 1\leqslant p_{l,r}\leqslant C_{n_{l,r}}}} (\mathbf{R}_{\Omega,\mathcal{S}})_{p_l}^{\lozenge n_l} \lozenge (\mathbf{R}_{\Omega,\mathcal{S}})_{p_r}^{\lozenge n_r} = \mathcal{R}_{\Omega,\mathcal{S}}.$$

□

3. The Fedosov construction for $\mathbb{C}^n$, $\mathbb{D}^n$ and $\mathbb{CP}^n$

The deformation quantization of the complex projective space $\mathbb{CP}^n$ is from a physical point of view an interesting example for the reduced phase space of an isotropic harmonic oscillator with $n + 1$ degrees of freedom.

3.1. The Fedosov differential for $\mathbb{C}^n$, $\mathbb{D}^n$ and $\mathbb{CP}^n$

Theorem 3.1.1. *For a Kaehler manifold of constant holomorphic sectional curvature C, Wick case of Fedosov construction and every formal power series $f(\lambda) \in \mathbb{C}[[\lambda]]$ the flat connection $\mathcal{D}_{-4\lambda f(\lambda)\omega}$ is*

$$\mathcal{D}_{-4\lambda f(\lambda)\omega} = \partial - \frac{1}{2\lambda}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sum_{l=0}^{n}(-1)^l\binom{n}{l}\sqrt{1-4\lambda(f(\lambda)+lC)}}{n!(2\lambda)^n}\right)g^{\vee n}\rho, \dots\right],$$

where ρ and $g^{\vee n}\rho$ denote $\rho = g_{i\bar{j}}(\mathrm{d}z^i \otimes \mathrm{d}\overline{z}^j - \mathrm{d}\overline{z}^j \otimes \mathrm{d}z^i)$ and $g^{\vee n}\rho = \underbrace{g \vee \dots \vee g}_{n} \cdot \rho$.

Proof. At first the super derivation δ can be written as a quasi inner derivation $\delta = \frac{1}{2\lambda}[\rho, \ \cdot\ ]$.

It is common sense [11] that the curvature tensor of a Kaehler manifold M of constant holomorphic sectional curvature C satisfies

$$R_{k\bar{l}i\bar{j}} = -C(g_{k\bar{l}}g_{i\bar{j}} + g_{k\bar{j}}g_{\bar{l}i}),$$

and M is holomorphically isometric to $\mathbb{C}^n$, $\mathbb{D}^n$ or the complex projective space $\mathbb{CP}^n$ if $C = 0$, $C < 0$ or $C > 0$ if M is simply connected.

Proposition 3.1.2. *If $\nabla_X\Omega = \nabla_X R = 0$ for all $X \in \Gamma^\infty(TM)$ the fixed point $\mathcal{R}_\Omega$ only has odd numbers of total degree* Deg $\geqslant 3$, *is likewise covariant constant and given by*

$$\mathcal{R}_\Omega = \sum_{n=1}^{\infty}\sum_{k=1}^{C_n}(\delta^{-1}(R+\Omega))_p^{(\frac{\mathrm{i}}{2\lambda}\delta^{-1}[\ \cdot\ ,\ \cdot\ ])n}.$$

The proof of this proposition is based on the two facts that ∇_X is a $\circ_\Pi$ derivation and commutes with δ^{-1}. These facts imply that the geometric series $(\mathrm{id} - \delta^{-1}\partial)^{-1}$ in theorem 2.2.2 break off to the identity.

That the even numbers of total degree Deg $\geqslant 4$ vanish is easy to prove by induction over the total degree, because $\delta^{-1}(\Omega + R)$ only has odd numbers of total degree Deg $\geqslant 3$.

For

$$\Omega_f := 2\lambda f(\lambda)\frac{1}{\mathrm{i}}g_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j = -4\lambda f(\lambda)\omega \in \lambda\Gamma^\infty(\Lambda^2 T^*M)[[\lambda]],$$

the proposition 3.1.2 and $R_{k\bar{l}i\bar{j}} = -C(g_{k\bar{l}}g_{i\bar{j}} + g_{k\bar{j}}g_{\bar{l}i})$ imply

$$\mathcal{R}_{\Omega_f} = \sum_{n=1}^{\infty}\sum_{k=1}^{C_n}\left(2\lambda f(\lambda)\frac{1}{2\mathrm{i}}g^{\vee 0}\rho + \frac{C}{2\mathrm{i}}g^{\vee 1}\rho\right)_p^{(\frac{\mathrm{i}}{2\lambda}\delta^{-1}[\ \cdot\ ,\ \cdot\ ])n}.$$

Only the Kaehler metric and insertion operators appear in this equation.

Proposition 3.1.3.

$$g^{\vee n_1}\rho\left(\frac{\mathrm{i}}{2\lambda}\delta^{-1}[\ \cdot\ ,\ \cdot\ ]\right)g^{\vee n_2}\rho = \mathrm{i}\sum_{n=0}^{\min\{n_1,n_2\}}(2\lambda)^n n!\binom{n_1}{n}\binom{n_2}{n}g^{\vee n_1+n_2-n}\rho.$$

With the identity

$$\frac{n!}{(n_1+n_2+2-n)}\left[\binom{n_2}{n}\binom{n_1}{n-1} + \binom{n_2}{n-1}\binom{n_1}{n} + \binom{n_1}{n-1}\binom{n_2}{n-1}\right]$$
$$= (n-1)!\binom{n_1}{n-1}\binom{n_2}{n-1},$$

for $n, n_1, n_2 \in \mathbb{N}$ the proof of the proposition is a straightforward calculation.

A projection of the proposition 3.1.3 on $\deg_\lambda = 0$ and calculation results in a formula for the classical part of $\mathcal{R}_{\Omega_f}$:

$$\mathcal{R}_{\Omega_f}{}^{2n+1}_{0} = \frac{C_n}{\mathrm{i}}\left(\frac{C}{2}\right)^n g^{\vee n}\rho.$$

The two propositions 3.1.2 and 3.1.3, taken together, imply

$$\mathcal{R}_{\Omega_f} = \sum_{n=0}^{\infty} C_n(2\lambda) g^{\vee n}\rho$$

with $C_n(2\lambda) \in \mathbb{C}[[\lambda]]$.

The explicit combinatoric of proposition 3.1.3 and the 'quadratic' fixed-point equation result in the recursion

$$C_0(2\lambda) = \frac{1 \pm \sqrt{1 - 4\lambda f(\lambda)}}{2\mathrm{i}},$$

and

$$C_{n\pm}(2\lambda) = \frac{-b_n \pm \sqrt{b_n^2 - 4\mathrm{i}(2\lambda)^n n! c_n}}{2\mathrm{i}(2\lambda)^n n!} \qquad \forall n \geqslant 1,$$

where b_n and c_n are

$$b_n = -1 + 2\mathrm{i}\sum_{n_1=0}^{n-1} C_{n_1}(2\lambda)(2\lambda)^{n_1} n_1! \binom{n}{n_1}$$

and

$$c_n = \mathrm{i}\sum_{n_1,n_2=1}^{n-1} C_{n_1}(2\lambda)C_{n_2}(2\lambda)(2\lambda)^{n_1+n_2-n}(n_1+n_2-n)!\binom{n_1}{n-n_2}\binom{n_2}{n-n_1}.$$

The equation

$$C_n(2\lambda) = (-1)^{n+1}\left(\frac{\sum_{l=0}^{n}(-1)^l\binom{n}{l}\sqrt{1-4\lambda(f(\lambda)+lC)}}{2\mathrm{i}n!(2\lambda)^n}\right) \qquad \forall n \geqslant 1$$

can be proved by induction.

At first the well-known fact

$$\sum_{m=0}^{n}(-1)^m m^k \binom{n}{m} = 0,$$

if $n \in \mathbb{N}^+$ and $k \in \mathbb{N}$ with $k < n$, is locking out principal Laurent parts of $C_n(2\lambda)$.

The so-called trinomial revision

$$\binom{h}{m}\binom{m}{k} = \binom{h}{k}\binom{h-k}{m-k}$$

and a lemma known as orthogonality

$$\sum_{m=m_1}^{m_2}(-1)^{m_1+m}\binom{m_2}{m}\binom{m}{m_1} = \delta_{m_1}^{m_2}$$

if $m_1, m_2 \in \mathbb{N}$, are helpful for computing in the inductive step.

Some substitutions and the orthogonality by induction over n show the following proposition, which is helpful for the calculation of the discriminant in the inductive step.

Proposition 3.1.4. *If* $l_1, l_2 \in \mathbb{N}$, $n \in \mathbb{N}^+$ *and* $0 \leqslant l_1, l_2 \leqslant n$,

$$\binom{n+1}{l_1} - \sum_{n_1=l_1}^{n}\sum_{n_2=l_2}^{n}(-1)^{n_1+n_2}\binom{n+1-l_2}{n+1-n_2}\binom{n_2}{n+1-n_1}\binom{n_1}{l_1} = (-1)^{n+l_1}\delta_{l_2}^{l_1}. \qquad \blacksquare$$

3.2. Fedosov star products for $\mathbb{C}^n$, $\mathbb{D}^n$ and $\mathbb{CP}^n$

Theorem 3.2.1. *For a formal power series $f(\lambda) \in \mathbb{C}[[\lambda]]$, Wick case and the closed two-form $\Omega_f := -4\lambda f(\lambda)\omega \in \lambda\Gamma^\infty(\Lambda^2 T^*M)[[\lambda]]$ the purely holomorphic part $\pi_z\tau_{\Omega_f}(h)$ and the purely anti-holomorphic part $\pi_{\overline{z}}\tau_{\Omega_f}(g)$ of the Fedosov–Taylor series on Kaehler manifolds of constant holomorphic sectional curvature C are*

$$\pi_z\tau_{\Omega_f}(h) = \sum_{n=0}^{\infty}\left(\prod_{l=0}^{n-1}\frac{1}{\sqrt{1-4\lambda(f(\lambda)+lC)}}\right)\left(\delta_z^{-1}\partial_z\right)^n h$$

and

$$\pi_{\overline{z}}\tau_{\Omega_f}(g) = \sum_{n=0}^{\infty}\left(\prod_{l=0}^{n-1}\frac{1}{\sqrt{1-4\lambda(f(\lambda)+lC)}}\right)\left(\delta_{\overline{z}}^{-1}\partial_{\overline{z}}\right)^n g.$$

Proof. According to Neumaier [16] the purely holomorphic part $\pi_z\tau_{\Omega_f}(h)$ and the purely anti-holomorphic part $\pi_{\overline{z}}\tau_{\Omega_f}(g)$ of the Fedosov–Taylor series satisfy independent fixed-point equations in special cases.

Lemma 3.2.2.

1. *If $\pi_z\mathcal{R}_{\text{Wick}} = 0$, then $\pi_z\tau_{\text{wick}}(h)\ \forall\ h \in C^\infty(M)[[\lambda]]$ satisfies*
$$\pi_z\tau_{\text{wick}}(h) = h + \delta_z^{-1}\left(\partial_z\pi_z\tau_{\text{wick}}(h) - \frac{\mathrm{i}}{\lambda}\pi_z(\pi_z\tau_{\text{wick}}(h)\circ_{\text{wick}}\mathcal{R}_{\text{Wick}})\right).$$
2. *If $\pi_{\overline{z}}\mathcal{R}_{\text{Wick}} = 0$, then $\pi_{\overline{z}}\tau_{\text{wick}}(g)\ \forall\ g \in C^\infty(M)[[\lambda]]$ satisfies the equation*
$$\pi_{\overline{z}}\tau_{\text{wick}}(g) = g + \delta_{\overline{z}}^{-1}\left(\partial_{\overline{z}}\pi_{\overline{z}}\tau_{\text{wick}}(g) + \frac{\mathrm{i}}{\lambda}\pi_{\overline{z}}(\mathcal{R}_{\text{Wick}}\circ_{\text{wick}}\pi_{\overline{z}}\tau_{\text{wick}}(g))\right).$$

The next proposition is a test of the fiberwise operations $\frac{\mathrm{i}}{\lambda}\delta_{\overline{z}}^{-1}\pi_{\overline{z}}[g^{\vee m}\rho,\ \cdot\ ]$ and $\frac{\mathrm{i}}{\lambda}\delta_z^{-1}\pi_z[g^{\vee m}\rho,\ \cdot\ ]$ on symmetric and purely holomorphic or purely anti-holomorphic elements.

Proposition 3.2.3.

1. *If $S \in \Gamma^\infty(\vee^s T^*M)$ with $\pi_z S = S$, then*
$$-\frac{\mathrm{i}}{\lambda}\delta_z^{-1}\pi_z(S\circ_\Pi g^{\vee m}\rho) = \mathrm{i}2(2\lambda)^m m!\binom{s-1}{m}S.$$
2. *If $\overline{S} \in \Gamma^\infty(\vee^s T^*M)$ with $\pi_{\overline{z}}\overline{S} = \overline{S}$, then*
$$+\frac{\mathrm{i}}{\lambda}\delta_{\overline{z}}^{-1}\pi_{\overline{z}}(g^{\vee m}\rho\circ_\Pi\overline{S}) = \mathrm{i}2(2\lambda)^m m!\binom{\overline{s}-1}{m}\overline{S}.$$

This proposition again follows by a straightforward calculation and implies

$$\pi_z\tau_{\Omega_f}(h) = \sum_{n=0}^{\infty}T_n(2\lambda)\left(\delta_z^{-1}\partial_z\right)^n h \quad \text{and} \quad \pi_{\overline{z}}\tau_{\Omega_f}(g) = \sum_{n=0}^{\infty}\overline{T}_n(2\lambda)\left(\delta_{\overline{z}}^{-1}\partial_{\overline{z}}\right)^n g.$$

The fixed-point equations of Neumaier 3.2.2, theorem 3.1.1, proposition 3.2.3 and the orthogonality, taken together, yield the recursion

$$T_o(2\lambda) = 1 = \overline{T}_0(2\lambda)$$

and

$$T_{n-1}(2\lambda) = \sqrt{1-4\lambda(f(\lambda)+(n-1)C)}\,T_n(2\lambda) = \overline{T}_{n-1}(2\lambda)$$

$\forall\ n \geqslant 1$. ∎

Neumaier showed in [16] that Fedosov's construction on a pseudo-Kaehler manifold is universal in the sense that it yields all star products of Wick type.

Another result in [16] is that the purely holomorphic part $\pi_z\tau_{\text{wick}}(h)$ and the purely anti-holomorphic part $\pi_{\bar{z}}\tau_{\text{wick}}(g)$ of the Fedosov–Taylor series are sufficient for calculating the Wick-type star product by

$$h \star_{\text{wick}} g = \sigma(\pi_z\tau_{\text{wick}}(h) \circ_{\text{wick}} \pi_{\bar{z}}\tau_{\text{wick}}(g)).$$

The explicit formulae of theorem 3.2.1 finally result in

Theorem 3.2.4. *For a Kaehler manifold M of constant holomorphic sectional curvature C and the closed two-form $\Omega_f := -4\lambda f(\lambda)\omega \in \lambda\Gamma^\infty(\Lambda^2 T^*M)[[\lambda]]$ the Fedosov Wick-type star product $\star_{\Omega_f} C^\infty(M)[[\lambda]] \times C^\infty(M)[[\lambda]] \to C^\infty(M)[[\lambda]]$ is*

$$h \star_{\Omega_f} g = \sum_{n=0}^{\infty} \frac{(2\lambda)^n}{n!} \left(\prod_{l=0}^{n-1} \frac{1}{1 - 4\lambda(f(\lambda) + lC)} \right) \mu \circ P^n \left(\left(\delta_z^{-1}\partial_z\right)^n h \otimes \left(\delta_{\bar{z}}^{-1}\partial_{\bar{z}}\right)^n g \right).$$

4. Conclusion

A non-recursive formula for the unique solution of Fedosov's 'quadratic' fixed-point equation in the general case was shown. This result is of interest because this fixed point is the corner stone in Fedosov's deformation quantization of symplectic manifolds and offers a method to formulate index theorems [9, 10, 17]. Also the proof could be interesting to deal with nonlinear, in some sense 'quadratic' fixed-point equations.

An adapted example for Kaehler manifolds of constant holomorphic sectional curvature C was considered in detail. This example is the first non-zero curvature example of a Fedosov construction.

A classical flat super derivation, well defined in the $\deg_s$-adic topology and with familiar shape in $-1 \leqslant \deg_s \leqslant 0$ likewise the classical part of the Fedosov super derivation, was introduced.

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